COALGEBRAS FOR A_∞- AND L_∞-ALGEBRAS

1. Coalgebras

Let $\mathbf{Vect}^{\mathbb{Z}}$ denote the category of $\mathbb{Z}$-graded vector spaces. Recall that morphisms in $\mathbf{Vect}^{\mathbb{Z}}$ are degree-preserving linear maps.

Definition 1.1. A *graded coalgebra* is an object $C \in \mathbf{Vect}^{\mathbb{Z}}$ together with a morphism $\Delta : C \to C \otimes C$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad\Delta\quad} & C \otimes C \\ \big\downarrow{\scriptstyle\Delta} & & \big\downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ C \otimes C & \xrightarrow{\mathrm{id}\otimes\Delta} & C \otimes C \otimes C \end{array}$$

is commutative. The morphism Δ is called a *coproduct*, and the fact that the above diagram commutes is referred to as *coassociativity.*

We define the iterated coproduct $\Delta^{(n)} : C \to C^{\otimes n}$ recursively by setting $\Delta^{(2)} := \Delta$ and $\Delta^{(n+1)} := (\Delta^{(n)} \otimes \mathrm{id})\Delta$ for $n \geq 2$. By convention, $\Delta^{(1)} := \mathrm{id}$. Coassociativity generalises to iterated coproducts in the following way:

Lemma 1.2. *For every $p, q \geq 1$, we have $\Delta^{(p+q)} = (\Delta^{(p)} \otimes \Delta^{(q)})\Delta$.*

We will often use Sweedler's summation convention, in which a sum

$$\Delta(x) = \sum_{1 \leq i \leq n} x_i^{(1)} \otimes x_i^{(2)}$$

is denoted $\Delta(x) = x^{(1)} \otimes x^{(2)}$, with the summation left implicit. More generally, we will write

$$\Delta^{(n)}(x) = x^{(1)} \otimes \cdots \otimes x^{(n)}.$$

So in this notation, Lemma 1.2 says that $x^{(1)} \otimes \cdots \otimes x^{(p+q)} = \Delta^{(p)}(x^{(1)}) \otimes \Delta^{(q)}(x^{(2)})$.

Definition 1.3. A *morphism of graded coalgebras* $f : (C, \Delta_C) \to (D, \Delta_D)$ is a morphism of the underlying graded vector spaces that makes the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad f\quad} & D \\ \big\downarrow{\scriptstyle\Delta_C} & & \big\downarrow{\scriptstyle\Delta_D} \\ C \otimes C & \xrightarrow{f\otimes f} & D \otimes D \end{array}$$

commute. In Sweedler notation,

$$f(x)^{(1)} \otimes f(x)^{(2)} = f(x^{(1)}) \otimes f(x^{(2)})$$

for all $x \in C$. We denote the category of graded coalgebras by $\mathbf{Coalg}_\circ^{\mathbb{Z}}$ (the subscript is supposed to indicate that we are considering *non-unital* coalgebras).

Note that if $f : (C, \Delta_C) \to (D, \Delta_D)$ is a morphism of graded coalgebras, then we have $f^{\otimes n}\Delta_C^{(n)} = \Delta_D^{(n)} f$ for all $n \geq 1$.

Recall that the category $\mathbf{Vect}^{\mathbb{Z}}$ is symmetric monoidal, with symmetries $\sigma : V \otimes W \to W \otimes V$ given by the *Koszul sign rule*

$$\sigma(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

Definition 1.4. A graded coalgebra (C, Δ) is *cocommutative* if $\Delta = \sigma\Delta$; that is, if

$$x^{(1)} \otimes x^{(2)} = (-1)^{|x^{(1)}||x^{(2)}|} x^{(2)} \otimes x^{(1)}$$

for all $x \in C$.

2. Two examples

In the section, we will define the two examples of graded coalgebras that will be important for us: the tensor coalgebra and the symmetric coalgebra. Fix a graded vector space $V \in \mathbf{Vect}^{\mathbb{Z}}$.

Definition 2.1. The *(reduced) tensor algebra* is defined by $T_\circ(V) := \bigoplus_{n\geq 1} V^{\otimes n}$.

Definition 2.2. Let I be the two-sided ideal of $T_\circ(V)$ generated by $v \otimes w - \sigma(v \otimes w)$, $v, w \in V$. The quotient $S_\circ(V) := T_\circ(V)/I$ is called the *(reduced) symmetric algebra.*

We will consider $T_\circ(V)$ and $S_\circ(V)$ as objects of $\mathbf{Vect}^{\mathbb{Z}}$, and not as algebras. We will use $\pi : T_\circ(V) \to S_\circ(V)$ to denote the canonical projection, and the image of a pure tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ under π will be denoted $v_1 v_2 \cdots v_n$.

Definition 2.3. Let $\Delta : T_\circ(V) \to T_\circ(V) \otimes T_\circ(V)$ be given by

$$\Delta(v_1 \otimes \cdots \otimes v_n) := \sum_{0<i<n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).$$

If we iterate Δ, we find the formula

$$\Delta^{(s)}(v_1 \otimes \cdots \otimes v_n) = \sum_{0<i_1<\cdots<i_{s-1}<n} (v_1 \otimes \cdots \otimes v_{i_1}) \otimes \cdots \otimes (v_{i_{s-1}+1} \otimes \cdots \otimes v_n). \quad (2.1)$$

In particular, Δ is coassociative. The coalgebra $T^c_\circ(V) := (T_\circ(V), \Delta)$ is called the *(reduced) tensor coalgebra cogenerated by V.*

Remark 2.4. Let $p : T^c_\circ(V) \to V$ denote the projection. It follows from Equation 2.1 that

$$p^{\otimes n}\Delta^{(n)}(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n$$

for all $v_1, \ldots, v_n \in V$.

To equip $S_\circ(V)$ with a coproduct, we need to introduce some notation. Let $v_1, \ldots, v_n$ be homogeneous vectors in V. Given a permutation $\sigma \in \Sigma_n$, we define the *Koszul sign* $\varepsilon(\sigma) = \pm 1$ by the relation

$$\varepsilon(\sigma) v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(n)} = v_1 v_2 \cdots v_n.$$

Now suppose $\mathcal{I} = I_1, \ldots, I_s$ is an ordered partition of $\{1, 2, \ldots, n\}$. Listing the elements of each I_j in order as $i_{j,1} < \cdots < i_{j,k_j}$, we write

$$v_{I_j} := v_{i_{j,1}} v_{i_{j,2}} \cdots v_{i_{j,k_j}}, \quad v_{\mathcal{I}} := v_{I_1} \cdots v_{I_s},$$

and we define the *Koszul sign* $\varepsilon(\mathcal{I})$ by $\varepsilon(\mathcal{I}) v_{\mathcal{I}} = v$, where $v = v_1 v_2 \cdots v_n$. (Note that $\mathcal{I}$ determines a permutation of $\{1, 2, \ldots, n\}$ in a natural way, and that $\varepsilon(\mathcal{I})$ is the Koszul sign of this permutation.) Finally, if I is a nonempty proper subset of $\{1, 2, \ldots, n\}$, then $\mathcal{I} = I, I^c$ is an ordered permutation of $\{1, 2, \ldots, n\}$, and we put $\varepsilon(I) := \varepsilon(\mathcal{I})$.

Definition 2.5. Let $\Delta : S_\circ(V) \to S_\circ(V) \otimes S_\circ(V)$ be given by

$$\Delta(v_1 v_2 \cdots v_n) := \sum_{\varnothing \subsetneq I \subsetneq \{1,\ldots,n\}} \varepsilon(I) v_I \otimes v_{I^c},$$

where $v_1, v_2, \ldots, v_n \in V$ are homogeneous vectors. Note that the coproduct Δ is well-defined and cocommutative. Iterating, we get

$$\Delta^{(s)}(v_1 v_2 \cdots v_n) = \sum_{\mathcal{I}=I_1,\ldots,I_s} \varepsilon(\mathcal{I}) v_{I_1} \otimes \cdots \otimes v_{I_s}, \tag{2.2}$$

where the sum is over all ordered partitions of $\{1, 2, \ldots, n\}$ with s blocks. In particular, Δ is coassociative. The cocommutative coalgebra $S^c_\circ(V) := (S_\circ(V), \Delta)$ is called the *(reduced) symmetric coalgebra cogenerated by V.*

Remark 2.6. If $p : S^c_\circ(V) \to V$ denotes the projection, then it follows from Equation 2.2 that

$$\frac{\pi}{n!} p^{\otimes n} \Delta^{(n)}(v_1 v_2 \cdots v_n) = v_1 v_2 \cdots v_n$$

for all $v_1, v_2, \ldots, v_n \in V$.

3. Cofreeness

The tensor and symmetric algebras $T_\circ(V)$ and $S_\circ(V)$ are free objects in their respective categories of non-unital algebras and non-unital commutative algebras. One would expect the coalgebras $T^c_\circ(V)$ and $S^c_\circ(V)$ to satisfy the dual universal property, called *cofreeness*. They do – but only once we restrict attention to the subcategory of $\mathbf{Coalg}^{\mathbb{Z}}_\circ$ consisting of the *conilpotent* coalgebras.

Definition 3.1. A *cogenerator* of a graded coalgebra (C, Δ) is a map of graded vector spaces $p : C \to V$ such that for every nonzero $x \in C$, there exists an integer $n \geq 1$ such that $p^{\otimes n} \Delta^{(n)}(x) \neq 0$.

Example 3.2. Any graded coalgebra C is trivially cogenerated by the identity map $\mathrm{id} : C \to C$.

Example 3.3. It follows from Remarks 2.4 and 2.6 that $T^c_\circ(V)$ and $S^c_\circ(V)$ are cogenerated by their projections onto V.

Proposition 3.4. *Let $p : D \to V$ be a cogenerator of a graded coalgebra (D, Δ_D). A morphism of graded coalgebras $f : (C, \Delta_C) \to (D, \Delta_D)$ is uniquely determined by the composition pf.*

Proof. It's enough to show $pf = 0 \implies f = 0$. Suppose $f \neq 0$, so that $f(x) \neq 0$ for some $x \in C$. Since p is a cogenerator of D, there exists an $n \geq 1$ such that $p^{\otimes n} \Delta^{(n)}(f(x)) \neq 0$. So we have

$$0 \neq p^{\otimes n} \Delta^{(n)}_D f = (pf)^{\otimes n} \Delta^{(n)}_C,$$

which implies $pf \neq 0$. □

Definition 3.5. A coalgebra (C, Δ) is called *conilpotent* if for each $x \in C$, we have $\Delta^{(n)} = 0$ for $n \gg 0$.

Example 3.6. The tensor coalgebra $T^c_\circ(V) = (T_\circ(V), \Delta)$ is conilpotent, since we have $\Delta^{(N)}(v_1 \otimes \cdots \otimes v_n) = 0$ as soon as $N > n$.

Example 3.7. The symmetric coalgebra $S^c_\circ(V) = (S_\circ(V), \Delta)$ is conilpotent, since we have $\Delta^{(N)}(v_1 \cdots v_n) = 0$ as soon as $N > n$.

Note that if (C, Δ) is a conilpotent coalgebra, the sum $\sum_{n\geq 1} \Delta^{(n)}(x)$ is finite for each $x \in C$, so the map

$$\sum_{n\geq 1} \Delta^{(n)} : C \to T_\circ(C)$$

is well-defined as a morphism of graded vector spaces.

Lemma 3.8. *Suppose $C \in \mathbf{Coalg}^{\mathbb{Z}}_\circ$ is conilpotent. Then $\sum_{n\geq 1} \Delta^{(n)} : C \to T^c_\circ(C)$ is a morphism of coalgebras.*

Proof. For every $x \in C$, we have

$$\begin{aligned}
\Delta_{T^c_\circ(C)} \sum_{n\geq 1} \Delta^{(n)}(x) &= \sum_{n\geq 1} \Delta_{T^c_\circ(C)}(x^{(1)} \otimes \cdots \otimes x^{(n)}) \\
&= \sum_{n\geq 1} \sum_{0<i<n} (x^{(1)} \otimes \cdots \otimes x^{(i)}) \otimes (x^{(i+1)} \otimes \cdots \otimes x^{(n)}) \\
&= \sum_{i\geq 1, j\geq 1} \Delta^{(i)}(x^{(1)}) \otimes \Delta^{(j)}(x^{(2)}) \\
&= \sum_{i\geq 1} \Delta^{(i)}(x^{(1)}) \otimes \sum_{j\geq 1} \Delta^{(j)}(x^{(2)}) \\
&= \Big(\sum_{i\geq 1} \Delta^{(i)} \otimes \sum_{j\geq 1} \Delta^{(j)} \Big) \Delta(x),
\end{aligned}$$

where we have used Lemma 1.2 in the third equality. □

If C is cocommutative, then a similar result holds for $S^c_\circ(C)$.

Lemma 3.9. *If $C \in \mathbf{Coalg}^{\mathbb{Z}}_\circ$ is conilpotent and cocommutative, then*

$$\sum_{n\geq 1} \frac{\pi}{n!} \Delta^{(n)} : C \to S^c_\circ(C)$$

is a morphism of coalgebras.

Proof. For every $x \in C$, we have

$$\begin{aligned}
\Delta_{S^c_\circ(C)} \sum_{n\geq 1} \frac{\pi}{n!} \Delta^{(n)}(x) &= \sum_{n\geq 1} \frac{1}{n!} \Delta_{S^c_\circ(C)}(x^{(1)} \cdots x^{(n)}) \\
&= \sum_{n\geq 1} \frac{1}{n!} \sum_{\varnothing \subsetneq I \subsetneq \{1,\dots,n\}} \varepsilon(I) x_I \otimes x_{I^c} \\
&= \sum_{n\geq 1} \frac{1}{n!} \sum_{0<i<n} \sum_{\substack{\varnothing \subsetneq I \subsetneq \{1,\dots,n\} \\ |I| = n}} \varepsilon(I) x_I \otimes x_{I^c} \\
&= \sum_{n\geq 1} \frac{1}{n!} \sum_{0<i<n} \binom{n}{i} (x^{(1)} \cdots x^{(i)}) \otimes (x^{(i+1)} \cdots x^{(n)}),
\end{aligned}$$

since C is cocommutative. On the other hand,

$$\begin{aligned}\Big(\sum_{i\geq 1}\frac{\pi}{i!}\Delta^{(i)}\otimes\sum_{j\geq 1}\frac{\pi}{j!}\Delta^{(j)}\Big)\Delta(x) &= \sum_{i\geq 1}\frac{\pi}{i!}\Delta^{(i)}(x^{(1)})\otimes\sum_{j\geq 1}\frac{\pi}{j!}\Delta^{(j)}(x^{(2)}) \\ &= \sum_{i,j\geq 1}\frac{\pi\otimes\pi}{i!j!}\Delta^{(i)}(x^{(1)})\otimes\Delta^{(j)}(x^{(2)}) \\ &= \sum_{n\geq 1}\sum_{0<i<n}\frac{\pi\otimes\pi}{i!(n-i)!}(x^{(1)}\otimes\cdots\otimes x^{(i)})\otimes(x^{(i+1)}\otimes\cdots\otimes x^{(n)}) \\ &= \sum_{n\geq 1}\sum_{0<i<n}\frac{1}{i!(n-i)!}(x^{(1)}\cdots x^{(i)})\otimes(x^{(i+1)}\cdots x^{(n)}).\end{aligned}$$

□

The following result says that $T^c_\circ(V)$ is cofree in the full subcategory of $\mathbf{Coalg}^{\mathbb{Z}}_\circ$ whose objects are the conilpotent coalgebras.

Proposition 3.10. *Let $C\in\mathbf{Coalg}^{\mathbb{Z}}_\circ$ be conilpotent, and let $V\in\mathbf{Vect}^{\mathbb{Z}}$. For any linear map $f : C\to V$, there exists a unique map of coalgebras $g : C\to T^c_\circ(V)$ making the diagram*

$$\begin{array}{ccc} & & T^c_\circ(V) \\ & \overset{\exists! g}{\nearrow} & \downarrow p \\ C & \xrightarrow{\ f\ } & V \end{array}$$

commute. Explicitly, we have

$$g(x)=\sum_{n\geq 1}f^{\otimes n}\Delta^{(n)}(x)=\sum_{n\geq 1}f(x^{(1)})\otimes\cdots\otimes f(x^{(n)})$$

for every $x\in C$.

Proof. Let g be the composition

$$C\xrightarrow{\sum_{n\geq 1}\Delta^{(n)}}T^c_\circ(C)\xrightarrow{T^c_\circ(f)}T^c_\circ(V),$$

which is a morphism of coalgebras by Lemma 3.8. Uniqueness follows from the fact that p is a cogenerator for $T^c_\circ(V)$. □

Similarly, $S^c_\circ(V)$ is cofree in the full subcategory of $\mathbf{Coalg}^{\mathbb{Z}}_\circ$ whose objects are the conilpotent cocommutative coalgebras.

Proposition 3.11. *Let $C\in\mathbf{Coalg}^{\mathbb{Z}}_\circ$ be conilpotent and cocommutative, and let $V\in\mathbf{Vect}^{\mathbb{Z}}$. For any linear map $f : C\to V$, there exists a unique map of coalgebras $g : C\to S^c_\circ(V)$ making the diagram*

$$\begin{array}{ccc} & & S^c_\circ(V) \\ & \overset{\exists! g}{\nearrow} & \downarrow p \\ C & \xrightarrow{\ f\ } & V \end{array}$$

commute. Explicitly, we have

$$g(x)=\sum_{n\geq 1}\frac{\pi}{n!}f^{\otimes n}\Delta^{(n)}(x)=\sum_{n\geq 1}\frac{1}{n!}f(x^{(1)})\cdots f(x^{(n)})$$

for every $x\in C$.

Proof. Let g be the composition

$$C \xrightarrow{\sum_{n\geq 1} \frac{\pi}{n!}\Delta^{(n)}} S^c_\circ(C) \xrightarrow{S^c_\circ(f)} S^c_\circ(V),$$

which is a morphism of coalgebras by Lemma 3.9. Uniqueness follows from the fact that p is a cogenerator for $S^c_\circ(V)$. □

4. Comodules

We introduce comodules, so that we can discuss coderivations in the next section.

Definition 4.1. If (C, Δ) is a graded coalgebra, then a *left C-comodule* is a vector space $N \in \mathbf{Vect}^{\mathbb{Z}}$ and a degree 0 linear map $\lambda : N \to C \otimes N$ making the diagram

$$\begin{array}{ccc} N & \xrightarrow{\quad\lambda\quad} & C\otimes N \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ C\otimes N & \xrightarrow{\mathrm{id}\otimes\lambda} & C\otimes C\otimes N \end{array} \tag{4.1}$$

commute. The map λ is called a *left coaction* of C on N.

Definition 4.2. If (C, Δ) is a graded coalgebra, then a *right C-comodule* is a vector space $N \in \mathbf{Vect}^{\mathbb{Z}}$ and a degree 0 linear map $\rho : N \to N \otimes C$ making the diagram

$$\begin{array}{ccc} N & \xrightarrow{\quad\rho\quad} & N\otimes C \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho\otimes\mathrm{id}} \\ N\otimes C & \xrightarrow{\mathrm{id}\otimes\Delta} & N\otimes C\otimes C \end{array} \tag{4.2}$$

commute. The map ρ is called a *right coaction* of C on N.

We will follow Sweedler's convention of denoting the sum

$$\lambda(x) = \sum_{1\leq i\leq n} x_i^{(1)} \otimes x_i^{(0)} \in C \otimes N$$

by $\lambda(x) = x^{(1)} \otimes x^{(0)}$. In this notation, the left coaction condition 4.1 reads

$$\Delta(x^{(1)}) \otimes x^{(0)} = x^{(1)} \otimes \lambda(x^{(0)}).$$

Similarly, we will write

$$\rho(x) = \sum_{1\leq i\leq n} x_i^{(0)} \otimes x_i^{(1)} \in N \otimes C$$

as $\rho(x) = x^{(0)} \otimes x^{(1)}$. The right coaction condition 4.2 reads

$$x^{(0)} \otimes \Delta(x^{(1)}) = \rho(x^{(0)}) \otimes x^{(1)}.$$

Definition 4.3. Let (C, Δ) be a graded coalgebra. A *C-bicomodule* is a vector space $N \in \mathbf{Vect}^{\mathbb{Z}}$ equipped with a left coaction $\lambda : N \to C \otimes N$ and a right coaction $\rho : N \to N \otimes C$, that are compatible in the sense that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\quad\rho\quad} & N\otimes C \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda\otimes\mathrm{id}} \\ C\otimes N & \xrightarrow{\mathrm{id}\otimes\rho} & C\otimes N\otimes C \end{array} \tag{4.3}$$

is commutative.

Example 4.4. Any coalgebra (C,Δ) is a bicomodule over itself, with $\lambda=\rho=\Delta$.

Suppose (N,λ,ρ) is a C-bicomodule, and $V\in\mathbf{Vect}^{\mathbb{Z}}$. Then $V\otimes N$ is again a C-bicomodule, with left coaction

$$V\otimes N\xrightarrow{\mathrm{id}_V\otimes\lambda}V\otimes C\otimes N\xrightarrow{\sigma\otimes\mathrm{id}_N}C\otimes V\otimes N$$

and right coaction

$$V\otimes N\xrightarrow{\mathrm{id}_V\otimes\rho}V\otimes N\otimes C.$$

Example 4.5. In particular, given $k\in\mathbb{Z}$, let $S^k\in\mathbf{Vect}^{\mathbb{Z}}$ be the vector space freely generated in degree k by a single element s^k. Then the suspension $S^k(N):=S^k\otimes N$ is a C-bicomodule with left coaction

$$s^kx\mapsto x_\lambda^{(1)}\otimes s^kx_\lambda^{(0)}(-1)^{k|x_\lambda^{(1)}|}$$

and right coaction

$$s^kx\mapsto s^kx_\rho^{(0)}\otimes x_\rho^{(1)}.$$

In the next section, we will use the following construction to understand coderivations as morphisms of coalgebras.

Lemma 4.6. *Let (C,Δ) be a graded coalgebra, and let N be a C-bicomodule with coactions $\lambda:N\to C\otimes N$ and $\rho:N\to N\otimes C$. Then $C\oplus N$ is a coalgebra, with coproduct $C\oplus N\to(C\oplus N)\otimes(C\oplus N)$ given by*

$$\Delta_{C\oplus N}(x,y)=\Delta(x)+\lambda(y)+\rho(y).$$

With this coalgebra structure, the inclusion $C\to C\oplus N$ and the projection $C\oplus N\to C$ are morphisms of coalgebras.

Proof. Write $\Delta_{C\oplus N}(x,y)$ as

$$x^{(1)}\otimes x^{(2)}+y_\lambda^{(1)}\otimes y_\lambda^{(0)}+y_\rho^{(0)}\otimes y_\rho^{(1)}.$$

If we apply $\Delta_{C\oplus N}\otimes\mathrm{id}$ to this expression, we get

$$\Delta(x^{(1)})\otimes x^{(2)}+\Delta(y_\lambda^{(1)})\otimes y_\lambda^{(0)}+\lambda(y_\rho^{(0)})\otimes y_\lambda^{(0)}+\rho(y_\rho^{(0)})\otimes y_\rho^{(1)},$$

while if we apply $\mathrm{id}\otimes\Delta_{C\oplus N}$, we obtain

$$x^{(1)}\otimes\Delta(x^{(2)})+y_\lambda^{(1)}\otimes\lambda(y_\lambda^{(0)})+y_\lambda^{(1)}\otimes\rho(y_\lambda^{(0)})+y_\rho^{(0)}\otimes\Delta(y_\rho^{(1)}).$$

The first terms are equal because Δ is coassociative, the second and fourth terms are equal because λ are ρ are coactions, while the third terms are equal because λ and ρ satisfy the compatibility condition 4.3 that makes N a bicomodule. □

We will need a formula for the iterations of the coproduct $\Delta_{C\oplus N}$. For $i,j\geq 0$, let $\Delta^{(i,j)}:N\to C^{\otimes i}\otimes N\otimes C^{\otimes j}$ be the diagonal of the following commutative square

$$\begin{array}{ccc} N & \xrightarrow{\lambda^i} & C^{\otimes i}\otimes N \\ \downarrow{\scriptstyle\rho^j} & & \downarrow{\scriptstyle\mathrm{id}_{C^{\otimes i}}\otimes\rho^j} \\ N\otimes C^{\otimes j} & \xrightarrow{\lambda^i\otimes\mathrm{id}_{C^{\otimes j}}} & C^{\otimes i}\otimes N\otimes C^{\otimes j}, \end{array}$$

where λ^i and ρ^j are defined recursively in the obvious manner.

Lemma 4.7. *We have*

$$\Delta^{(n)}_{C\oplus N}(x,y) = \Delta^{(n)}(x) + \sum_{i+j=n-1} \Delta^{(i,j)}(y),$$

for all $n \geq 1$.

Proof. We use induction on n. The statement clearly holds for $n=1$ and $n=2$, so suppose the result is true for some $n \geq 2$. We have

$$\Delta^{(n+1)}_{C\oplus N}(x,y) = \left(\Delta^{(n)}_{C\oplus N} \otimes \mathrm{id}\right)\Delta(x) + \left(\Delta^{(n)}_{C\oplus N} \otimes \mathrm{id}\right)\lambda(y) + \left(\Delta^{(n)}_{C\oplus N} \otimes \mathrm{id}\right)\rho(y).$$

Since $\operatorname{im}\Delta \subseteq C\otimes C \subseteq (C\oplus N)\otimes(C\oplus N)$, the induction hypothesis implies

$$\left(\Delta^{(n)}_{C\oplus N} \otimes \mathrm{id}\right)\Delta = \left(\Delta^{(n)} \otimes \mathrm{id}\right)\Delta = \Delta^{(n+1)}.$$

Similarly, since $\operatorname{im}\lambda \subseteq C\otimes N \subseteq (C\oplus N)\otimes(C\oplus N)$, we have

$$\left(\Delta^{(n)}_{C\oplus N} \otimes \mathrm{id}\right)\lambda = \left(\Delta^{(n)} \otimes \mathrm{id}\right)\lambda = \left(\mathrm{id}\otimes\lambda^{n-1}\right)\lambda = \lambda^n,$$

where we have repeatedly used the commutativity of Diagram 4.1. Finally, since $\operatorname{im}\rho \subseteq N\otimes C$, we have

$$\begin{aligned}
\left(\Delta^{(n)}_{C\oplus N} \otimes \mathrm{id}\right)\rho &= \sum_{i+j=n-1} \left(\Delta^{(i,j)} \otimes \mathrm{id}\right)\rho \\
&= \sum_{i+j=n-1} \left(\lambda^i \otimes \mathrm{id}_{C^{\otimes j+1}}\right)(\rho^j\otimes \mathrm{id})\rho \\
&= \sum_{i+j=n-1} \left(\lambda^i \otimes \mathrm{id}_{C^{\otimes j+1}}\right)\rho^{j+1} \\
&= \sum_{i+j=n-1} \Delta^{(i,j+1)}.
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
\Delta^{(n+1)}_{C\oplus N}(x,y) &= \Delta^{(n+1)}(x) + \lambda^n(y) + \sum_{i+j=n-1} \Delta^{(i,j+1)}(y) \\
&= \Delta^{(n+1)}(x) + \Delta^{(n,0)}(y) + \sum_{\substack{i+j=n \\ j\geq 1}} \Delta^{(i,j)}(y) \\
&= \Delta^{(n+1)}(x) + \sum_{i+j=n} \Delta^{(i,j)}(y),
\end{aligned}$$

which is the required formula. □

5. Coderivations

Given $V, W \in \mathbf{Vect}^{\mathbb{Z}}$, we write $\mathrm{Hom}^k(V,W)$ for the vector space of linear maps from V to W of degree k. The internal Hom of the category $\mathbf{Vect}^{\mathbb{Z}}$ is

$$\mathrm{Hom}^\bullet(V,W) := \oplus_{k\in\mathbb{Z}} \mathrm{Hom}^k(V,W).$$

Definition 5.1. Let (C,Δ) be a graded coalgebra and let (N,λ,ρ) be a C-bicomodule. We say that a linear map $D \in \mathrm{Hom}^k(N,C)$ is a *degree k coderivation of C with domain N* (or simply a *coderivation*) if it satisfies the coLeibniz rule

$$\Delta D = (D\otimes \mathrm{id})\rho + (\mathrm{id}\otimes D)\lambda.$$

In Sweedler notation, this reads

$$D(x)^{(1)} \otimes D(x)^{(2)} = D(x_\rho^{(0)}) \otimes x_\rho^{(1)} + (-1)^{k|x_\lambda^{(1)}|} x_\lambda^{(1)} \otimes D(x_\lambda^{(0)})$$

for every $x \in N$. We write $\mathrm{Coder}^k(N, C)$ for the vector space of degree k coderivations of C with domain N. If $N = C$, we write $\mathrm{Coder}^k(C) := \mathrm{Coder}^k(C, C)$.

The following proposition allows us to think of a degree 0 coderivation as a morphism of coalgebras.

Proposition 5.2. *Let C be a graded coalgebra and N a C-bicomodule. A linear map $D \in \mathrm{Hom}^0(N, C)$ is a coderivation if and only if the map $g : C \oplus N \to C$ defined by putting $g(x, y) = x + D(y)$ is a morphism of coalgebras, where $C \oplus N$ has the coalgebra structure of Lemma 4.6.*

Proof. Suppose $(x, y) \in C \oplus N$. Then

$$\begin{aligned}(g \otimes g)\Delta_{C\oplus N}(x, y) &= (g \otimes g)\big(x^{(1)} \otimes x^{(2)} + y_\rho^{(0)} \otimes y_\rho^{(1)} + y_\lambda^{(1)} \otimes y_\lambda^{(0)}\big) \\ &= x^{(1)} \otimes x^{(2)} + D(y_\rho^{(0)}) \otimes y_\rho^{(1)} + y_\lambda^{(1)} \otimes D(y_\lambda^{(0)}).\end{aligned}$$

On the other hand,

$$\Delta g(x, y) = \Delta(x + D(y)) = x^{(1)} \otimes x^{(2)} + D(y)^{(1)} \otimes D(y)^{(2)}.$$

Therefore,

$$(g \otimes g)\Delta_{C\oplus N} - \Delta g = (D \otimes \mathrm{id})\rho + (\mathrm{id} \otimes D)\lambda - \Delta D.$$

□

Corollary 5.3. *If $p : C \to V$ is a cogenerator, then a coderivation $D \in \mathrm{Hom}^k(N, C)$ is uniquely determined by the composition pD.*

Proof. Follows from Proposition 3.4. □

Recall from Example 4.5 that the suspension $S^k N$ of a bicomodule N is again a bicomodule. Any degree k coderivation (with domain N) can be turned into a degree 0 coderivation (with domain $S^k N$) in the following way.

Lemma 5.4. *There is an isomorphism of vector spaces*

$$\mathrm{Coder}^k(N, C) \xrightarrow{\simeq} \mathrm{Coder}^0(S^k N, C),$$

which assigns to a coderivation $D \in \mathrm{Coder}^k(N, C)$ the coderivation $D' \in \mathrm{Coder}^0(S^k N, C)$ obtained by setting $D'(s^k x) := D(x)$ for all $x \in N$.

We are finally in a position to understand the coderivations of $T^c_\circ(V)$ and $S^c_\circ(V)$.

Theorem 5.5. *Let $V \in \mathbf{Vect}^{\mathbb{Z}}$. Then every degree k linear map $q \in \mathrm{Hom}^k(T^c_\circ(V), V)$ coextends to a unique coderivation $Q \in \mathrm{Coder}^k(T^c_\circ(V))$. Explicitly, Q is given by the formula*

$$Q(v_1 \otimes \cdots \otimes v_n) = \sum_{l=1}^{n} \sum_{i=0}^{n-l} (-1)^{k \sum_{j=1}^{i} |v_j|} v_1 \otimes \cdots \otimes v_i \otimes q(v_{i+1} \otimes \cdots \otimes v_{i+l}) \otimes \cdots \otimes v_n.$$

Proof. Let $N = S^k T^c_\circ(V)$, and consider the coalgebra $T^c_\circ(V) \oplus N$ of Lemma 4.6. Define a map $f : T^c_\circ(V) \oplus N \to V$ by

$$f(x, s^k y) := p(x) + q(y),$$

where $p : T^c_\circ(V) \to V$ is the canonical projection. The coalgebra $T^c_\circ(V) \oplus N$ is clearly conilpotent, so by Proposition 3.10, there is a unique morphism of coalgebras $g : T^c_\circ(V) \oplus N \to T^c_\circ(V)$ coextending f. Now the inclusion $i_1 : T^c_\circ(V) \to T^c_\circ(V) \oplus N$ is a morphism of coalgebras, so $gi_1 : T^c_\circ(V) \to T^c_\circ(V)$ a morphism of coalgebras. Moreover, we have $pgi_1 = fi_1 = p$, so the uniqueness part of Proposition 3.10 implies $gi_1 = \mathrm{id}$. Therefore, by Proposition 5.2, g is of the form

$$g(x, s^k y) = x + Q'(s^k y),$$

with $Q' \in \mathrm{Coder}^0(N, T^c_\circ(V))$. Let Q be given by $Q(y) := Q'(s^k y)$. Then Lemma 5.4 implies $Q \in \mathrm{Coder}^k(T^c_\circ(V))$, and we have

$$pQ(y) = pg(0, s^k y) = f(0, s^k y) = q(y),$$

so Q is a coextension of q. Uniqueness of Q follows from Corollary 5.3.

It remains to compute Q on an input $v_1 \otimes \cdots \otimes v_n \in T^c_\circ(V)$. According to Example 4.5, the right coaction of $T^c_\circ(V)$ on $N = S^k T^c_\circ(V)$ is given by the formula

$$\rho(s^k v_1 \otimes \cdots \otimes v_n) = \sum_{0<i<n} s^k(v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n),$$

while the left coaction is given by

$$\lambda(s^k v_1 \otimes \cdots \otimes v_n) = \sum_{0<i<n} (-1)^{k \sum_{j=1}^{i} |v_j|} (v_1 \otimes \cdots \otimes v_i) \otimes s^k(v_{i+1} \otimes \cdots \otimes v_n).$$

A formula for $\Delta^{(a,b)} = (\lambda^a \otimes \mathrm{id})\rho^b : N \to T^c_\circ(V)^{\otimes a} \otimes N \otimes T^c_\circ(V)^{\otimes b}$ is

$$\Delta^{(a,b)}(s^k v_1 \otimes \cdots \otimes v_n) = \sum_{0<i_1<\cdots<i_{a+b}<n} (-1)^{k \sum_{j=1}^{i_a} |v_j|} (v_1 \otimes \cdots \otimes v_{i_1}) \otimes \cdots \otimes \\ s^k(v_{i_a+1} \otimes \cdots \otimes v_{i_{a+1}}) \otimes \cdots \otimes (v_{i_{a+b}+1} \otimes \cdots \otimes v_n).$$

By Proposition 3.10 and Lemma 4.7,

$$g(x, s^k y) = \sum_n \Delta^{(n)}(x) + \sum_{i+j=n-1} \Delta^{(i,j)}(y)$$

□